

Equilibrium boundary conditions, dynamic vacuum energy, and the Big Bang

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Abstract

The near-zero value of the cosmological constant Λ in an equilibrium context may be due to the existence of a self-tuning relativistic vacuum variable q . Here, a cosmological nonequilibrium context is considered with a corresponding time-dependent cosmological parameter $\Lambda(t)$ or vacuum energy density $\rho_V(t)$. A specific model of a closed Friedmann–Robertson–Walker universe is presented, which is determined by equilibrium boundary conditions at one instant of time ($t = t_{\text{eq}}$) and a particular form of vacuum-energy dynamics ($d\rho_V/dt \propto \rho_M$). This homogeneous and isotropic model has a standard Big Bang phase at early times ($t \ll t_{\text{eq}}$) and reproduces the main characteristics of the present universe ($t = t_0 < t_{\text{eq}}$).

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I. INTRODUCTION

It has been argued that the gravitating vacuum energy density ρ_V or cosmological constant Λ [1, 2, 3, 4, 5] vanishes in a perfect quantum vacuum, provided that this vacuum can be considered to be a self-sustained medium at zero external pressure and that there exists a new type of conserved microscopic variable q which self-adjusts so as to give vanishing internal pressure [6]. As the perfect quantum vacuum is Lorentz invariant (see, e.g., Refs. [7, 8] for bounds on Lorentz violation in the photon sector), this vacuum “charge” q must be of an entirely new type, different from known conserved charges such as baryon number minus lepton number, $B - L$. The detailed microscopic theory is, of course, unknown, but two examples of possible theories with such a vacuum variable q have been given in Ref. [6].

For the perfect Lorentz-invariant quantum vacuum, the vacuum variable q is constant over the whole of spacetime. The previous discussion then applies to an equilibrium situation and describes what may be called the “statics of dark energy.” Two outstanding questions are, first, how the equilibrium argument relates to the observed expanding universe and, second, which physical principle governs the “dynamics of dark energy.” Obviously, these are profound questions and the present article can only hope to provide a small step towards a possible solution. In fact, the first question is temporarily replaced by the following restricted question:

Is it possible at all to relate equilibrium boundary conditions for $\rho_V(t_{\text{eq}})$ to an expanding universe which matches the observations, even if we are free to choose the type of vacuum-energy dynamics, $d\rho_V/dt \neq 0$?

In mathematical terms, we are after an “existence proof” for this type of model universe, which has equilibrium boundary conditions setting the numerical value of the vacuum energy density ρ_V at one moment in time (here, coordinate time $t = t_{\text{eq}} \equiv 0$).

It turns out to be rather difficult to construct such an existence proof, but, in the end, we have been able to find one suitable class of universes. The main lesson we will learn from this exercise is the necessity of some form of “instability” of the imperfect quantum vacuum (for the case considered, Lorentz invariance is perturbed by the presence of thermal matter and spatial curvature) and we will get an idea of what type of instability would be required to reproduce the Universe as observed [2, 3, 4, 5]. In a way, our goal is to find the “Kepler laws” of the accelerating universe, leaving the underlying physics to future generations.

The outline of this article is as follows. The topic of dynamic vacuum energy density in the context of q -theory is introduced in Sec. II. A closed Friedmann–Robertson–Walker (FRW) universe with a generalized *Ansatz* for the vacuum-energy dynamics is then discussed in Sec. III. The corresponding numerical solution is presented in Sec. IV (related results for the case of vanishing vacuum energy density are relegated to the Appendix). Final comments are given in Sec. V.

II. DYNAMIC VACUUM ENERGY DENSITY FROM q -THEORY

A. Gravitational action with four-form and scalar fields

The crucial issue is the exchange of energy between the deep vacuum (described in part by the conserved microscopic variable q) and the low-energy degrees of freedom corresponding to the physics of the standard model and general relativity. The detailed microscopic theory is unknown, but we can try to seek guidance from the concrete four-form theory considered in Ref. [6].

This particular theory, coupled to low-energy matter, is defined by the action [6, 9, 10]

$$S = \int_{\mathbb{R}^4} d^4x \sqrt{-g} \left(\frac{R}{16\pi G_N} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} - \epsilon(F) \left(1 + \frac{1}{2} \phi^2/M^2 \right) \right), \quad (2.1a)$$

$$F^2 \equiv -\frac{1}{24} F_{\mu\nu\rho\sigma} F_{\alpha\beta\gamma\delta} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma}, \quad (2.1b)$$

$$F_{\mu\nu\rho\sigma} \equiv \nabla_{[\mu} A_{\nu\rho\sigma]}, \quad (2.1c)$$

where $R(x)$ is the Ricci curvature scalar from the metric $g_{\mu\nu}(x)$, $F_{\mu\nu\rho\sigma}(x)$ the four-form field strength of a three-form gauge field $A_{\nu\rho\sigma}(x)$, and ∇_μ the covariant derivative. In addition, the microscopic energy density $\epsilon(F)$ is taken to be an arbitrary function of F and the low-energy matter field $\phi(x)$ a real scalar field with coupling constant $1/M^2$ to $\epsilon(F)$. Here, and in the rest of this section, we use natural units with $\hbar = c = 1$.

The variational principle applied to action (2.1a) results in three field equations, a generalized Maxwell equation for the $F_{\mu\nu\rho\sigma}$ field, a generalized Klein–Gordon equation for the ϕ field, and the standard Einstein equation for the $g_{\mu\nu}$ field with an energy-momentum tensor $T_{\mu\nu}$ from both $F_{\mu\nu\rho\sigma}$ and ϕ fields.

B. Vacuum energy density in a flat FRW universe

In order to solve the field equations from the model action (2.1), the following *Ansatz* can be used: a spatially-flat ($k = 0$) Friedmann–Robertson–Walker metric [2], a Levi–Civita-type four-form field [6, 9, 10], and a homogenous scalar field. Specifically, the *Ansatz* fields are given by

$$g_{\mu\nu}(x) = \text{diag}(+1, -a(t)^2, -a(t)^2, -a(t)^2) , \quad (2.2a)$$

$$F_{\mu\nu\rho\sigma}(x) = q(t) |a(t)|^3 e_{\mu\nu\rho\sigma} , \quad (2.2b)$$

$$\phi(x) = \phi(t) , \quad (2.2c)$$

with scale factor $a(t)$ and totally antisymmetric Levi–Civita symbol $e_{\mu\nu\rho\sigma}$. The generalized Maxwell equation reduces then to

$$\frac{\dot{q}}{q} = -\chi_V q \epsilon' \frac{\phi \dot{\phi}}{M^2 + \phi^2/2} , \quad (2.3)$$

in terms of the vacuum compressibility [6]

$$\chi_V \equiv (q^2 \epsilon'')^{-1} , \quad (2.4)$$

with the prime standing for differentiation with respect to the vacuum variable q and the overdot for differentiation with respect to the cosmic time coordinate t .

With the *Ansatz* fields (2.2), there are two contributions to the energy-momentum tensor $T_{\mu\nu}$ in the Einstein field equation. The first contribution to $T_{\mu\nu}$ is proportional to the metric, $T_{\mu\nu}^V = \rho_V g_{\mu\nu}$, and corresponds to a vacuum energy density

$$\rho_V = \left(\epsilon(q) - q \epsilon'(q) \right) \equiv \tilde{\epsilon}(q) , \quad (2.5)$$

which equals the previous result $\tilde{\epsilon}(q)$ from Ref. [6]. In the following, it will be assumed that the equilibrium value q_c is such that $\tilde{\epsilon}(q_c) > 0$ and $\chi_V(q_c) > 0$, where the value q_c [different from the value q_0 for Minkowski spacetime with $\tilde{\epsilon}(q_0) = 0$] may result from some type of perturbation as discussed in Ref. [6].

The second contribution to $T_{\mu\nu}$ corresponds to the energy-momentum tensor of a comoving perfect fluid with energy density and pressure [3, 4, 5] given by

$$\rho_M = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \tilde{\mu}^2 \phi^2 , \quad P_M = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \tilde{\mu}^2 \phi^2 , \quad (2.6)$$

in terms of the effective mass square

$$\tilde{\mu}^2(q) \equiv \tilde{\epsilon}(q)/M^2, \quad (2.7)$$

which is positive as long as $\tilde{\epsilon}(q)$ and M^2 are. At the equilibrium value $q = q_c$, define $\tilde{\mu}_c^2 \equiv \tilde{\mu}^2(q_c)$.

For later, it turns out to be useful to introduce already the following equations of state for matter and vacuum:

$$P_M = w_M \rho_M, \quad P_V = w_V \rho_V = -\rho_V. \quad (2.8)$$

The matter equation-of-state parameter can be time dependent, $w_M = w_M(t)$, as it is simply the ratio of the two terms in (2.6). But the vacuum equation-of-state parameter is strictly constant, $w_V = -1$, as the corresponding energy-momentum tensor [6] is given by $T_{\mu\nu}^V = \rho_V g_{\mu\nu}$. This different behavior traces back to the special nature of the four-form field without propagating degrees of freedom [9, 10] and to the fact that there are no derivative terms of F in the original action (2.1a).

C. Energy exchange between vacuum and matter

The structure of the vacuum energy density (2.5) from the simple model considered allows us to say something concrete about the energy exchange between vacuum and matter. From the reduced Maxwell equation (2.3), the time derivative of (2.5) is given by

$$\dot{\rho}_V = \tilde{\epsilon}' \dot{q} = -\hat{\chi} \tilde{\epsilon} \frac{\phi \dot{\phi}}{M^2 + \phi^2/2}, \quad (2.9)$$

for the dimensionless quantity

$$\hat{\chi} \equiv (q \tilde{\epsilon}' / \tilde{\epsilon}) (q \epsilon') (q^2 \epsilon'')^{-1}, \quad (2.10)$$

whose absolute value may be of order 1 for generic $\epsilon(q)$. Considering small field values $\phi^2 \ll M^2$ (see below), the final expression reads

$$\dot{\rho}_V(t) = \text{sgn}[-\phi(t) \dot{\phi}(t)] \hat{\mu}(t) |\tilde{\mu}_c| \sqrt{1 - w_M^2(t)} \rho_M(t), \quad (2.11)$$

for a further dimensionless quantity

$$\hat{\mu}(t) \equiv \hat{\chi}(t) |\tilde{\mu}(q(t))| / |\tilde{\mu}_c|, \quad (2.12)$$

which can also be assumed to be of order 1, as long as $q(t)$ remains close to q_c [recall the definition of $\tilde{\mu}_c^2$ a few lines below (2.7)]. For completeness, the sign function used in (2.11) has $\text{sgn}[x] \equiv x/|x|$ for $x \neq 0$ and $\text{sgn}[0] \equiv 0$.

At this moment, a brief comment on the energy scales involved may be appropriate. The microscopic energy density $\epsilon(F)$ can be assumed to be of order $(E_{\text{Planck}})^4$, with $E_{\text{Planck}} \equiv \sqrt{\hbar c^5/G_N} \approx 1.22 \times 10^{28}$ eV. In addition, if (2.11) is to play a role in the energy balance and evolution of the present universe (see Sec. IV), one requires the following order of magnitudes [5]: $\rho_V \sim \rho_M \sim (10^{-3} \text{ eV})^4$ and $\tilde{\mu}_c \sim 10^{-33}$ eV. With the $\tilde{\mu}_c^2$ definition below (2.7), this gives $M \sim \sqrt{\rho_V}/(10^{-33} \text{ eV}) \sim 10^{27}$ eV, which corresponds to a Planckian energy scale. Of course, it remains to be seen if such a toy-model version (2.1a) of q -theory is relevant to the ultimate microscopic theory.

To summarize, result (2.11) describes the change of vacuum energy density due to non-trivial matter dynamics ($\dot{\phi} \neq 0$) and nonzero vacuum compressibility ($\chi_V > 0$). However, (2.11) holds only for matter described by a single real scalar field ϕ and the flat ($k = 0$) FRW universe. More importantly, the dimensionless microscopic function $\hat{\mu}(t)$ is not at all known, even if it can be expected to be of order unity. In the following, we, therefore, work with an *Ansatz* for $\dot{\rho}_V$ which is kept as general as possible but still proportional to ρ_M .

III. CLOSED FRW UNIVERSE AND NONTRIVIAL VACUUM DYNAMICS

A. Standard dynamics

The spatially flat ($k = 0$) FRW universe does not have an obvious time for equilibrium boundary conditions, apart from the limiting case with $a(t) \rightarrow \infty$ and $\rho_M(t) \rightarrow 0$ as $t \rightarrow \infty$. For this reason, we turn to the closed ($k = 1$) FRW universe [2] with metric

$$g_{00}(x) = 1, \quad g_{m0}(x) = 0, \quad g_{mn}(x) = -a^2(t) \hat{g}_{mn}(x), \quad (3.1)$$

in terms of the standard metric \hat{g}_{mn} of a unit 3-sphere for spatial indices $m, n = 1, 2, 3$. The scale factor $a(t)$ now corresponds to the radius of the closed universe and, as is well-known, can have a stationary point at a finite value of $a(t)$.

Henceforth, we discuss only the dynamics of classical relativity and use units with $c = 8\pi G_N/3 = 1$, unless stated otherwise. Note, however, that the boundary conditions to be presented in Sec. IIIB may rely implicitly on quantum mechanics, as does the vacuum instability to be discussed in Secs. IIIC and IIID.

The dynamics of the standard closed ($k = 1$) FRW universe [2] is governed by the 00-component of the Einstein equation,

$$\ddot{a}/a = -(4\pi G_N/3) (\rho_{\text{total}} + 3 P_{\text{total}}) = (8\pi G_N/3) \left(\rho_V - \frac{1}{2} (1 + 3 w_M) \rho_M \right), \quad (3.2)$$

the energy-conservation equation,

$$(\dot{\rho}_V + \dot{\rho}_M) = -3 (\dot{a}/a) (1 + w_M) \rho_M, \quad (3.3)$$

and a trivial vacuum-energy equation,

$$\dot{\rho}_V = 0, \quad (3.4)$$

which corresponds to the case of a genuine cosmological constant (spacetime-independent vacuum energy density). Equations (3.2) and (3.3) have been derived for equation-of-state (EOS) parameters

$$w_M = \text{const}, \quad w_V = -1. \quad (3.5)$$

Here, the matter EOS parameter has been assumed to be time independent, but this assumption can be relaxed later. The vacuum EOS parameter w_V is to remain fixed to the value -1 , which is the case for q -theory [6] as mentioned a few lines below (2.8).

Recall that, combined with energy conservation (3.3), the first-order Friedmann equation,

$$(\dot{a}/a)^2 = (8\pi G_N/3) (\rho_V + \rho_M) - k/a^2 \Big|_{k=1}, \quad (3.6)$$

is equivalent [2] to the second-order Einstein equation (3.2), at least, for appropriate boundary conditions.

B. Static Einstein universe from equilibrium boundary conditions

For the task outlined in Sec. I (obtaining an “existence proof”), the *static* Einstein universe [1] suffices, as it corresponds to an equilibrium state with constant radius a and constant energy densities ρ_V and ρ_M . This static closed universe can simply be taken as the starting point of the discussion in Sec. III C, but it is also possible to give an argument for the two conditions that single out this particular universe from other closed FRW universes.

In fact, the following two conditions can be seen to nullify the right-hand sides of the differential equations (3.2), (3.3), and (3.6). The first condition makes sure that the expansion

momentarily stops ($\dot{a}/a = 0$) at the equilibrium point $t_{\text{eq}} \equiv 0$:

$$(8\pi G_{\text{N}}/3) \left(\rho_{\text{V}}(t_{\text{eq}}) + \rho_{\text{M}}(t_{\text{eq}}) \right) = k a(t_{\text{eq}})^{-2} \Big|_{k=1}, \quad (3.7)$$

with the gravitational coupling constant G_{N} and the dimensionless curvature parameter k shown temporarily. The second condition makes the acceleration or deceleration vanish ($\ddot{a}/a = 0$) at the equilibrium point $t_{\text{eq}} \equiv 0$:

$$\rho_{\text{V}}(t_{\text{eq}}) = w_{\text{M}} \rho_{\text{M}}(t_{\text{eq}}) + \frac{1}{2} (1 + w_{\text{M}}) \rho_{\text{M}}(t_{\text{eq}}), \quad (3.8)$$

where, strictly speaking, w_{M} stands for $w_{\text{M}}(t_{\text{eq}})$, but, here, w_{M} has been assumed constant. Clearly, condition (3.7) does not require a nonzero value of the vacuum energy density, whereas (3.8) does, provided the model universe contains matter. Historically, this was indeed the reason for Einstein [1] to introduce his original cosmological constant, as he was aiming for a static universe.

Let us briefly comment on a possible interpretation of this last condition, which, in this context, was first discussed by Volovik [11]. The first term on the right-hand side of (3.8) corresponds to the flat-spacetime condition $\rho_{\text{V}} = P_{\text{M}} = w_{\text{M}} \rho_{\text{M}}$ from pressure equilibrium $P_{\text{V}} + P_{\text{M}} = P_{\text{ext}} = 0$ and the vacuum equation of state $P_{\text{V}} = -\rho_{\text{V}}$. See Ref. [6] for an extensive discussion of this flat-spacetime result, which traces back to a Gibbs–Duhem-type equation derived in q -theory. The second term on the right-hand side of (3.8) describes the gravitational effects, even though Newton’s gravitational constant G_{N} does not appear explicitly in the final result [note that G_{N} does enter condition (3.7) explicitly]. Specifically, the complete relation (3.8) follows from the two conditions $P_{\text{V}} + P_{\text{M}} + P_{\text{grav}} = 0$ and $G_{\text{N}}(\rho_{\text{V}} + \rho_{\text{M}} + \rho_{\text{grav}}) = 0$ for an effective gravitational equation of state $P_{\text{grav}} = -(1/3) \rho_{\text{grav}}$. See Sec. 7 of Ref. [11] for further discussion of this curvature contribution to (3.8).

Conditions (3.7) and (3.8) are simply boundary conditions for the cosmological equations (3.2)–(3.6). But this last condition (3.8) can also be argued from thermodynamic principles [11] and, for the quantum vacuum as discussed in Ref. [6] and Sec. II here, would have a naturally small vacuum energy density by the self-adjustment of the vacuum variable q . The self-adjustment may, in fact, be the result from a very long phase in the “life” of the model universe, as will be discussed in Sec. III D.

C. Nonstatic universe from vacuum instability

Given the boundary conditions (3.7) and (3.8) and given the task of somehow recovering the observed (expanding!) Universe, the problem is to get *away* from the static Einstein universe [1] with $a(t) = a(0)$, $\rho_V(t) = \rho_V(0)$, and $\rho_M(t) = \rho_M(0)$. It appears that the only way to achieve this is to consider either a modification of gravity (e.g., a modified Einstein field equation as studied in Ref. [12]) or a new type of instability of the imperfect quantum vacuum. The present article follows the second approach.

Specifically, we assume that (3.4) is replaced by the following *Ansatz* for the time variation of the vacuum energy density:

$$\dot{\rho}_V(t) = \Gamma_{VM} \gamma(t) \rho_M(t), \quad (3.9)$$

with a dimensionless functional $\gamma[a(t)/a_{eq}] \equiv \gamma(t)$, normalized by $\gamma[1] = 1$, and a new fundamental decay constant $\Gamma_{VM} > 0$ [here, quantum mechanics may enter if, for example, $\Gamma_{VM} \propto mc^2/\hbar$ for a mass scale m , as in (2.11) from the simple version of q -theory discussed in Sec. II]. As mentioned before, the origin of (3.9) needs to be explained by the detailed microphysics, but, here, we take a purely phenomenological (“Keplerian”) approach and simply assume a particular form for $\dot{\rho}_V$. Remark that, for $w_M = 0$, ρ_M in (3.9) can be interpreted as corresponding to the cold-dark-matter energy density from observational cosmology [5], with the baryonic contribution neglected.

Equations (3.2), (3.3), and (3.9) with boundary conditions (3.7) and (3.8) can then be solved numerically to give $a(t)$, $\rho_M(t)$, and $\rho_V(t)$. As we intend to take equilibrium-point boundary conditions also for the standard case with $\rho_V(t) = 0$ [some relevant results are given in the Appendix], we use the second-order 00-component Einstein equation (3.2) instead of the first-order Friedmann equation (3.6). As mentioned before, it is a well-known fact [2] that, with appropriate boundary conditions, the differential equations (3.2) and (3.6) are equivalent when combined with the energy-conservation equation (3.3). Incidentally, the 11-component of the Einstein equation is also satisfied, as are the 22 and 33 components by isotropy.

D. Additional remarks

In this subsection, further remarks are presented on the background and context of the vacuum-instability *Ansatz* (3.9). These remarks are, however, not essential for the rest of

this article and can be skipped in a first reading.

To start, three technical remarks on *Ansatz* (3.9) are in order:

1. $\dot{\rho}_V$ vanishes if $\rho_M = 0$, but ρ_V can still be nonzero, so that a de-Sitter universe remains a possible solution for the case of $\rho_M = 0$ and $\rho_V = \text{const} > 0$;
2. $\dot{\rho}_V$ does not necessarily vanish if $\dot{a}/a = 0$ and, in particular, $\dot{\rho}_V$ does not vanish at $t = t_{\text{eq}} \equiv 0$, so that the model universe can get away from the static Einstein universe;
3. time-reversal invariance around t_{eq} is manifestly broken if $\gamma(t)$ is continuous at $t = t_{\text{eq}}$.

Note that *Ansatz* (3.9) resembles Eq. (8) of Ref. [13] with $\Gamma_{VV} \equiv 1/\tau \neq 0$ and Eq. (3) of Ref. [14], but differs by points 1 and 2, respectively. Observe also that point 1 holds precisely for result (2.11) derived from the toy-model version of q -theory in Sec. II.

From point 2 above and with $\Gamma_{VM} \gamma(t_{\text{eq}}) > 0$, there is, in principle, the possibility of having a “Big Bang” with $a(t_{\text{BB}}) = 0$ at $t_{\text{BB}} < t_{\text{eq}}$. Remark that the direction of the coordinate time t has no direct physical meaning for the homogenous models considered here, as the physical “arrow-of-time” appears to be related to the “growth” of inhomogeneities “originating” from a smooth Big Bang [15] (see also Ref. [16] for an explicit T-violation mechanism in a closed nonisotropic universe).

From point 3, there is the possibility that, even with a Big Bang at $t_{\text{BB}} < t_{\text{eq}}$, the model universe does *not* return to vanishing 3-volume for $t > t_{\text{eq}}$. One possible scenario is that the function $\gamma(t)$ has a discontinuous jump to $\gamma(t) = 0$ for $t > t_{\text{eq}}$ and that the homogeneous model universe is static for $t \in [t_{\text{eq}}, \infty)$. There would then be an infinitely long equilibrium phase which makes the discussion of an self-adjusting vacuum variable q quite natural [6] (the vacuum variable q may also play a crucial role for the stability issue; see Sec. II C of Ref. [6]). Considering the coordinate time t to “start” at a large positive value and to “run” in the negative direction, the nonstatic universe then “takes off” at $t \equiv 0$ due to the sudden onset of instability, leading to a “Big Bang” for an appropriate behavior of $\gamma(t)$ at $t \leq 0$, as will be discussed in the next section. This fluctuation scenario resembles, in a way, earlier discussions [17] on the tunneling origin of the nonstatic universe (around $a \sim 0$), but our fluctuation “starts at the other end,” that is, $a \sim a_{\text{eq}}$.

As a final remark, we emphasize that the model considered in the present article is based on the Gibbs–Duhem-type condition (3.8) of the static Einstein universe, which may arise from the self-adjustment of a conserved relativistic variable q characterizing the microscopic

quantum vacuum [6]. In this respect, the closed model universe presented here is complementary to the model of a scalar field evolving towards an attractor (see, e.g., Ref. [18] and references therein), as this type of scalar model does not solve the quantum-mechanical cosmological constant problem of why ρ_V vanishes in Minkowski spacetime without fine tuning. The general analysis of an evolving scalar field may, indeed, turn out to be relevant for an accurate description of the present universe with $\rho_V \sim \rho_M \ll E_{\text{Planck}}^4$, especially if the effective scalar field can be related to a conserved microscopic variable q . In the present article, however, we do not consider the dynamics of q or other microscopic fields and use, instead, the simple phenomenological *Ansatz* (3.9).

IV. NONSTANDARD CLOSED FRW UNIVERSE

A. Specific γ Ansatz

As explained in the Introduction, our goal is relatively modest: to find at least one functional $\gamma[a(t)/a_{\text{eq}}]$ so that Eqs. (3.2), (3.3) and (3.9), with boundary conditions (3.7) and (3.8) can produce a solution which more or less reproduces our known Universe (see, e.g., Refs. [5, 19, 20, 21, 22] and references therein), which is spatially flat to a high degree of precision and approximately consists of 75 % “dark energy” and 25 % matter (primarily nonbaryonic “cold dark matter”).

With three coupled nonlinear ordinary differential equations (ODEs), this modest goal is surprisingly difficult to reach. Still, we have been successful by first considering the *inverse* problem which consists of the following two steps: (i) to find, given a more or less reasonable $a_{\text{designer}}(t)$, which densities $\rho_M(t)$ and $\rho_V(t)$ are required; (ii) to determine, by differentiation of the $\rho_V(t)$ from the first step, the required $\Gamma_{VM} \gamma(t)$ from (3.9).

Inspired by these “designer-universe” results, we make the following *Ansatz* for the (dimensionless) vacuum-dynamics functional:

$$\gamma[\alpha(t)] = \alpha^2 f_{c_1}(1 - \alpha) \sin(c_2 \pi \alpha) + \alpha f_{c_1}(\alpha) \left(\frac{(c_3)^{1/3}}{(c_3)^{1/3} + |1 - \alpha|^{1/3}} \right)^4, \quad (4.1a)$$

$$f_c(x) \equiv x^6 (1 + c^6) / (x^6 + c^6), \quad (4.1b)$$

with $\alpha(t) \equiv a(t)/a_{\text{eq}}$ restricted to the range $[0, 1]$ and numerical coefficients $c_n > 0$. Roughly speaking, this *Ansatz* for $\gamma(t)$ consists of a sharply-peaked positive term modulated to be effective just below $a = a_{\text{eq}}$ and a term proportional to a^3 modulated to be effective near

$a = 0$. A nonzero value of $\gamma(t_{\text{eq}})$ will be seen to be needed to get a nonstatic universe and the behavior $\gamma \propto a^3$ near $a = 0$ will be seen to allow for a finite limiting value of $\rho_V(a)$ by compensating the divergent $w_M = 0$ behavior $\rho_M \propto 1/a^3$ on the right-hand side of (3.9).

B. Numerical solution

A concrete model universe can be obtained by taking the following numerical values (in units with $8\pi G_N/3 = c = 1$) for the boundary conditions at $t = t_{\text{eq}} \equiv 0$ and the model parameters (namely, the matter EOS parameter w_M , the vacuum decay constant Γ_{VM} , and the *Ansatz* coefficients c_n):

$$\begin{pmatrix} a(0) \\ \rho_M(0) \\ \rho_V(0) \\ w_M \\ \Gamma_{VM} \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 2/300 \\ 1/300 \\ 0 \\ 50 \\ 1/5 \\ 9/4 \\ 1/15 \end{pmatrix}, \quad (4.2)$$

with the implicit equilibrium condition $\dot{a}/a = 0$ at $t = 0$ from the Friedmann equation (3.6). Remark that, if, for example, the value of the curvature radius $a(0)$ is fixed, the values of the energy densities $\rho_M(0)$ and $\rho_V(0)$ are determined by the equilibrium conditions (3.7) and (3.8), for given w_M .

The numerical solution of the coupled ODEs (3.2), (3.3), and (3.9) with *Ansatz* (4.1) and boundary conditions (4.2) is given in Fig. 1. [It has been verified explicitly that this numerical solution also solves the Friedmann equation (3.6).] Observe that $\Gamma_{VM} = 0$ would give a static Einstein universe with $a(t) = a(0)$, $\rho_V(t) = \rho_V(0)$, and $\rho_M(t) = \rho_M(0)$ at the values indicated by the heavy dots in Fig. 1. As explained in Sec. IIIC and IIID, we have appealed to a new type of “instability” of the imperfect quantum vacuum with $\Gamma_{VM} \gamma(t_{\text{eq}}) > 0$ in order to get away from this static universe (for time coordinate t starting at a value 0 and running in the negative direction, so that ρ_V decreases initially).

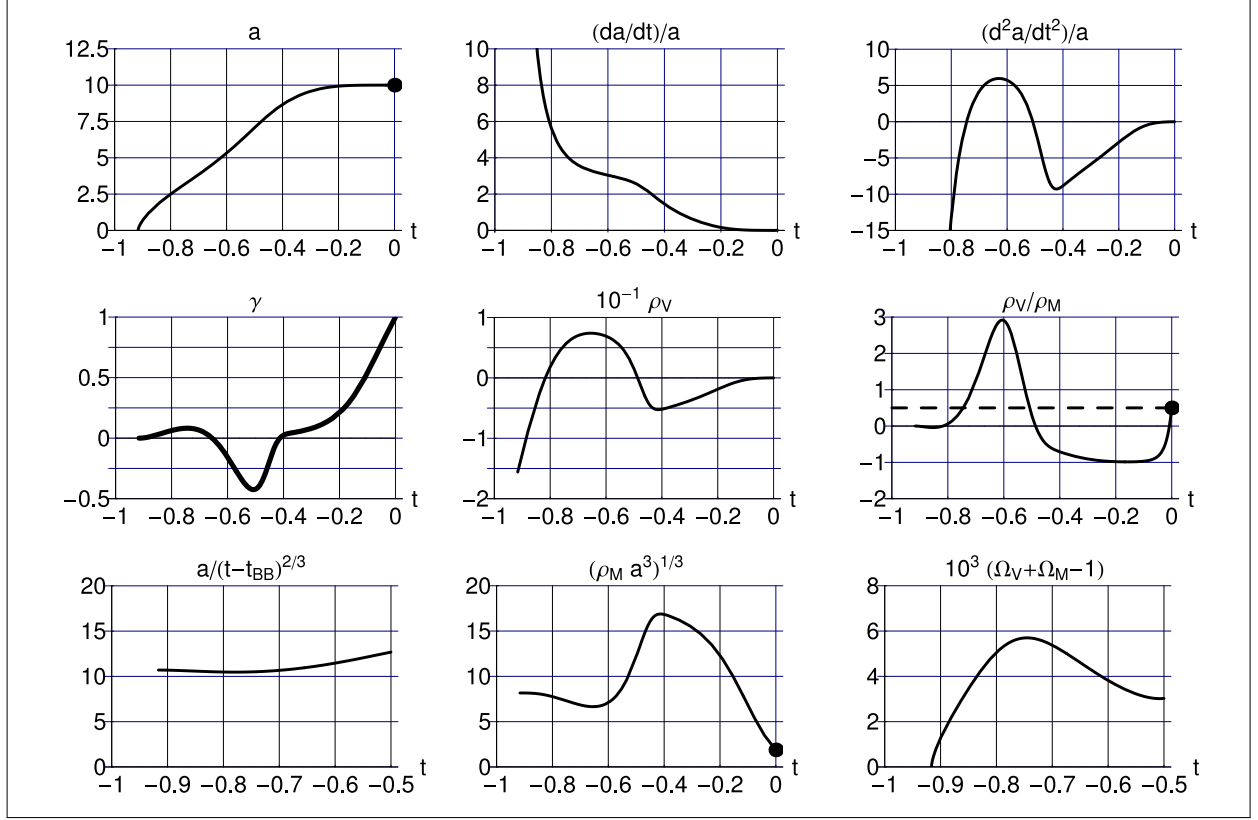


FIG. 1: Closed FRW universe with pressureless matter ($w_M = 0$) and dynamic vacuum energy ($w_V = -1$), for boundary conditions (4.2) in units with $8\pi G_N/3 = c = 1$. The assumed behavior of the vacuum-energy dynamics is given by (3.9) with the functional $\gamma[a(t)/a_{\text{eq}}]$ from (4.1). The three nonzero equilibrium boundary conditions on a , ρ_M , and ρ_V at $t = t_{\text{eq}} \equiv 0$ are indicated by the heavy dots (only shown if clearly different from zero) and the functional γ (with particular values for the numerical coefficients c_1 , c_2 , and c_3) is indicated by the heavy curve in the left-most panel of the middle row. Moreover, the vacuum decay constant Γ_{VM} has been set to 50 and this relatively large value explains the rapid change of ρ_V/ρ_M near $t = 0$. The scale factor $a(t)$ vanishes at $t = t_{\text{BB}} = -0.91636$ and the expansion of the model universe is accelerated ($\ddot{a}/a > 0$) if $\rho_V/\rho_M > 1/2$, as indicated by the dashed curve in the right-most panel of the middle row.

C. Big Bang and present universe recovered

Turning to the detailed model results of Fig. 1, the “Big Bang” with $a(t_{\text{BB}}) = 0$ would occur at coordinate time $t = t_{\text{BB}} = -0.91636$, which differs by 1 order of magnitude from the result without vacuum energy in the Appendix. Still, approximately the same behavior for $t \downarrow t_{\text{BB}}$ is observed for both model universes, namely, a scale factor vanishing as $a(t) \propto$

$(t - t_{\text{BB}})^{2/3}$ and a matter energy density diverging as $\rho_{\text{M}} \propto a^{-3}$, with the vacuum energy density $\rho_{\text{V}}(t)$ in Fig. 1 approaching a finite value at $t = t_{\text{BB}}$ [cf. the last sentence of Sec. IV A].

The “present universe” with density ratio $\rho_{\text{V}}/\rho_{\text{M}} \approx 2.75$ (close to the WMAP-5yr mean value from Table 1 in Ref. [22] for $h = 0.70$) would approximately correspond to the time $t = t_0 = -0.5842$ in Fig. 1 (choosing the latest time of two possible times, which both happen to be close to the maximum of $\rho_{\text{V}}/\rho_{\text{M}}$). The model values of the present universe are then

$$\begin{pmatrix} t \\ t - t_{\text{BB}} \\ a \\ \dot{a}/a \\ \rho_{\text{M}} \\ \rho_{\text{V}} \\ \rho_{\text{V}}/\rho_{\text{M}} \\ \Omega_{\text{V}} + \Omega_{\text{M}} \end{pmatrix} = \begin{pmatrix} -0.5842 \\ 0.3322 \\ 5.582 \\ 2.985 \\ 2.384 \\ 6.557 \\ 2.750 \\ 1.004 \end{pmatrix}, \quad (4.3)$$

where Ω_{X} is the energy density ρ_{X} relative to the critical density $\rho_{\text{crit}} \equiv (\dot{a}/a)^2$ in units with $8\pi G_{\text{N}}/3 = c = 1$

By identifying the calculated value $\dot{a}/a = 2.985$ with the measured value [23] of the Hubble constant $H_0 \equiv h/(9.78 \times 10^9 \text{ yr}) \approx 0.70/(9.78 \text{ Gyr})$, the present age of the model universe $t_0 - t_{\text{BB}} \approx 0.3322$ becomes

$$\tau_0 \approx 13.85 (0.70/h) \text{ Gyr}. \quad (4.4)$$

Similarly, the present radius of the model universe $a_0 \approx 5.582$ becomes of the order of $2 \times 10^{11} \text{ lyr}$, significantly larger than the present particle horizon. It is far from trivial that more or less reasonable values for $\rho_{\text{V}0}/\rho_{\text{M}0}$, $\Omega_{\text{V}0} + \Omega_{\text{M}0}$, and τ_0 can be produced at all in our approach.

The equilibrium time $t_{\text{eq}} - t_{\text{BB}} \approx 0.91636$ of the model universe corresponds to $\tau_{\text{eq}} \approx 38.22 \text{ Gyr}$, but there need not be a Big Crunch at even later times because of the possible lack of time-reversal invariance. In fact, there may be a very long static phase with $a(t) = a(0)$ for $t \geq 0$, if $\gamma(t) = 0$ for positive times t . This possibility has already been discussed in the penultimate paragraph of Sec. III D.

With the measured photon temperature $T_{\gamma 0} \approx 3 \text{ K}$ and the model value $a_0 \approx 6$, the matter EOS parameter w_{M} must change to a value $1/3$ for $0 \leq a \lesssim 6 \times 10^{-3}$ (relativistic

matter being dominant for $T \gtrsim 3000$ K), in order to recover the standard nucleosynthesis of the very early universe [2, 4]. In order to maintain a finite ρ_v value as $a \downarrow 0$, the *Ansatz* (4.1) can have the factor α^2 in the first term on the right-hand side changed to α^3 , for example, so that $\gamma \propto a^4$ for very small values of a .

As to the phenomenology of $\gamma[a(t)/a_{\text{eq}}]$, we clearly recognize three phases in Fig. 1, where $\gamma(t)$ is positive, negative, and again positive as the time coordinate t moves away from $t_{\text{eq}} = 0$ in the negative direction. (Other structures of γ are not excluded *a priori*, but the one found suffices for the present discussion.) The resulting behavior of $\rho_v(t)$ from (3.9) is shown in the figure panel to the right of the one of $\gamma(t)$. The fact that there is energy exchange between vacuum and matter is demonstrated by the nonconstant behavior of $\rho_M a^3$ as shown by the middle panel of the bottom row of Fig. 1 (compare with the results in the Appendix). Note that $\rho_v(t_{\text{BB}})$ need not be negative, as different Γ_{VM} values and coefficients c_n in (4.1) can give positive $\rho_v(t_{\text{BB}})$ values of order 1 or perhaps $\rho_v(t_{\text{BB}}) = 0$. Different Γ_{VM} values and coefficients c_n can also give a ρ_v/ρ_M peak value larger than 3, but it may be difficult to keep the “present age” of the model universe at the value (4.4) and to prevent it from dropping to a significantly lower value.

From the approximate linearity of $a(t)$ up to the “present value” $t_0 \approx -0.5842$ in Fig. 1, it is possible to relate the time coordinate t just below t_0 to the redshift z used by observational cosmology through the approximate relation $1 + z \approx (t_0 - t_{\text{BB}})/(t - t_{\text{BB}})$. Then, a coordinate time $t = -0.75$ would correspond to a redshift $z \approx 1$ and the model vacuum energy density $\rho_v(z)$ from Fig. 1 is seen to be more or less constant for redshifts z between 0 and 1. In fact, if future observations can measure $\rho_v(z)$ and $\rho_M(z)$ up to $z \approx 3$ (see, e.g., Ref. [24] for theoretical considerations), this would indirectly constrain the behavior of $\Gamma_{\text{VM}} \gamma(t)$ for $t \in (t_{\text{BB}}, t_0]$. These observations can perhaps also constrain $\Gamma_{\text{VM}} \gamma(t)$ over the *whole* range $[t_{\text{BB}}, t_{\text{eq}}]$ if there are effective two-boundary conditions such as $\rho_v(t_{\text{BB}}) = 0$ and $\rho_v(t_{\text{eq}}) \neq 0$ from the underlying microscopic physics (possibly with a new mechanism of T and CPT violation [15, 16]).

V. DISCUSSION

By way of summary, we list the main features of the particular closed model universe of Sec. IV and Fig. 1:

1. a Gibbs–Duhem-type boundary condition (3.8) at $t = t_{\text{eq}}$ with a finite vacuum energy

density $\rho_V(t_{\text{eq}}) = (1/2) \rho_M(t_{\text{eq}})$ for matter with equation-of-state parameter $w_M = 0$ [this particular value for $\rho_V(t_{\text{eq}})$ may result from the self-tuning of a conserved microscopic variable q to an equilibrium value q_c];

2. finite $|\rho_V(t)|$ within a factor of order 10^3 from the value set at $t = t_{\text{eq}}$ (see point 1);
3. a standard Big Bang phase at $t \sim t_{\text{BB}} < t_{\text{eq}}$ having $a(t) \propto (t - t_{\text{BB}})^{2/3}$ for $w_M = 0$, matter energy density $\rho_M \propto a^{-3}$, and energy density ratio $\rho_V/\rho_M \rightarrow 0$ for $t \downarrow t_{\text{BB}}$;
4. an accelerating phase for “present times,” with ρ_V/ρ_M of order 1 and an approximately flat 3-geometry.

This model universe constitutes the “existence proof” announced in Sec. I. Points 1 and 2 suggest, moreover, that a nonvanishing vacuum energy density $\rho_V(t)$ relevant to cosmology may not require fine-tuning by factors of order $(E_{\text{Planck}}/10^{-3} \text{ eV})^4 \sim 10^{124}$ due to the self-adjustment [6] of the vacuum variable q in an equilibrium phase $t \geq t_{\text{eq}}$.

Still, it remains to be explained theoretically that the fundamental vacuum-dynamics constant $c/\Gamma_{\text{VM}} \approx 1 \times 10^9 \text{ yr} \approx \hbar c/(2 \times 10^{-32} \text{ eV})$ is of the order of the length scale $a(t_{\text{eq}}) \approx 4 \times 10^{11} \text{ yr}$ of the equilibrium model universe. [As mentioned before, the single quantity $a(t_{\text{eq}})$ determines the two other quantities $\rho_M(t_{\text{eq}})$ and $\rho_V(t_{\text{eq}})$ from conditions (3.7) and (3.8) for given value of w_M .] The theoretical explanation of this very small energy scale $\hbar \Gamma_{\text{VM}} \approx 2 \times 10^{-32} \text{ eV}$ would, most likely, trace back to the detailed microphysics, perhaps along the lines of the simple version of q -theory discussed in Sec. II C. Inversely, there is the possibility that observational cosmology, by measuring the time dependence of the vacuum energy density, can provide information on the microscopic structure of the quantum vacuum.

ACKNOWLEDGMENTS

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NOTE ADDED

The q -theory approach [6] to dynamical vacuum energy density in cosmology has been elaborated in two recent articles [25, 26].

APPENDIX A: STANDARD CLOSED FRW UNIVERSE

In this appendix, a standard closed FRW universe [2] is reviewed which has the same extremal radius as the nonstandard universe discussed in Sec. IV. Specifically, the boundary conditions at $t = t_{\max} \equiv 0$ and the matter equation-of-state parameter w_M are (in units with $8\pi G_N/3 = c = 1$):

$$\begin{pmatrix} a(0) \\ \rho_M(0) \\ \rho_V(0) \\ w_M \end{pmatrix} = \begin{pmatrix} 10 \\ 1/100 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{A1})$$

Note that boundary conditions (A1) imply $\dot{a}/a = 0$ at $t = 0$ by the Friedmann equation (3.6).

The corresponding numerical solution of the differential equations (3.2), (3.3), and (3.4) is displayed in Fig. 2. The analytic solution, in terms of an auxiliary angle $\theta \in [0, 2\pi]$, is

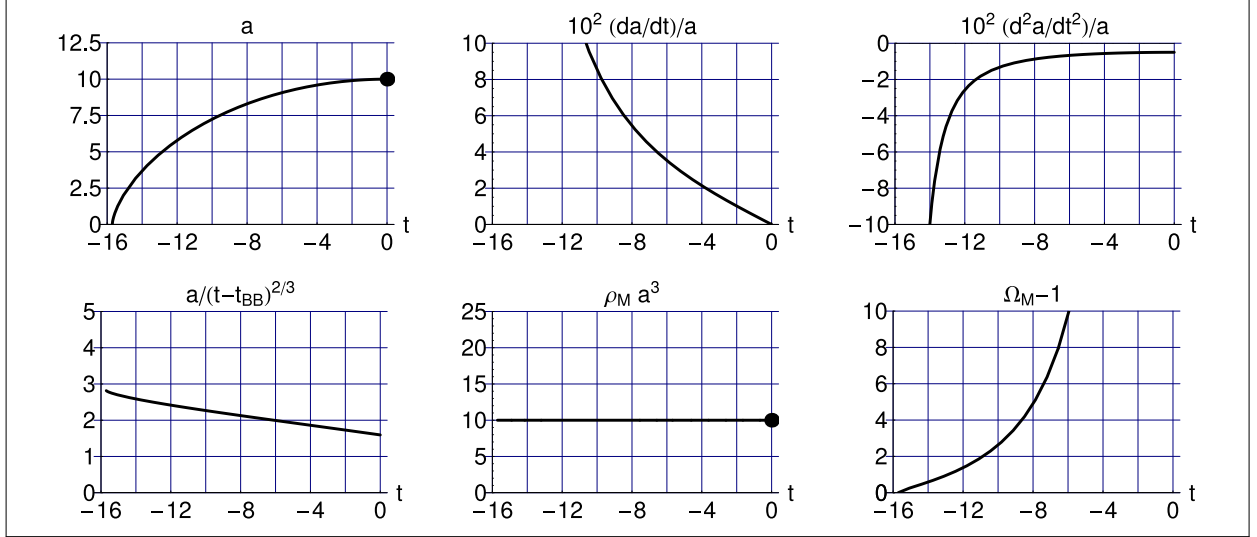


FIG. 2: Closed FRW universe with pressureless-matter energy density $\rho_M(t)$ and vanishing vacuum energy density $\rho_V(t)$ [not displayed], for boundary conditions (A1) in units with $8\pi G_N/3 = c = 1$. On the first row are shown the scale factor $a(t)$ and various derivatives, \dot{a}/a and \ddot{a}/a . On the second row are shown $a(t)$ scaled by a fractional power of the elapsed time since $t_{\text{BB}} = -5\pi \approx -15.71$ where $a(t)$ vanishes, the matter energy density ρ_M multiplied by a^3 , and the matter-density parameter $\Omega_M \equiv \rho_M/\rho_{\text{crit}}$ defined in terms of the critical density $\rho_{\text{crit}} \equiv (\dot{a}/a)^2$. The two boundary conditions on a and ρ_M at $t = t_{\max} \equiv 0$ are indicated by heavy dots.

given by [2, 3]

$$a = a_{\max} \sin^2(\theta/2), \quad (\text{A2a})$$

$$\rho_{\text{M}}(a) = a_{\max}/a^3, \quad \rho_{\text{V}}(a) = 0, \quad (\text{A2b})$$

$$t = (\theta - \sin \theta - \pi) a_{\max}/2, \quad (\text{A2c})$$

with $a_{\max} = 10$ from (A1). The time-symmetric solution (A2a) has Big Bang coordinate time $t_{\text{BB}} = -\pi a_{\max}/2$ and Big Crunch coordinate time $t_{\text{BC}} = +\pi a_{\max}/2$. For $t \downarrow t_{\text{BB}}$, the behavior of $a(t)$ approaches that of the flat ($k = 0$) FRW universe, $a(t) \propto (t - t_{\text{BB}})^{2/3}$.

These results for a standard closed FRW universe without vacuum energy serve as benchmark for those of the nonstandard universe discussed in Secs. III and IV. For example, the comparison of the three top-row panels in Figs. 1 and 2 highlights the different behavior at the stationary point $t = 0$. Similarly, the time dependence or time independence of $\rho_{\text{M}} a^3$ in the respective middle bottom-row panel indicates the presence or absence of energy exchange between vacuum and matter.

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- [1] A. Einstein, “Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie,” Sitzungsber. Preuss. Akad. Wiss. 1917, Phys.–Math. Klasse, p. 142; reprinted in: *The Collected Papers of Albert Einstein, Vol. 6, The Berlin Years: Writings 1914–1917*, edited by A. Kox et al., (Princeton University Press, Princeton, 1996), Doc. 43; translated as: “Cosmological considerations on the general theory of relativity,” in: *The Principle of Relativity*, edited by H.A. Lorentz *et al.* (Dover Publ., New York, 1952), Chap. IX.
 - [2] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
 - [3] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
 - [4] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
 - [5] S. Weinberg, *Cosmology* (Oxford University Press, Oxford, England, 2008).
 - [6] F.R. Klinkhamer and G.E. Volovik, “Self-tuning vacuum variable and cosmological constant,” Phys. Rev. D **77**, 085015 (2008), arXiv:0711.3170.
 - [7] A. Kostelecký and M. Mewes, “Signals for Lorentz violation in electrodynamics,” Phys. Rev. D **66**, 056005 (2002), arXiv:hep-ph/0205211.

- [8] (a) F.R. Klinkhamer and M. Risse, “Ultrahigh-energy cosmic-ray bounds on nonbirefringent modified-Maxwell theory,” *Phys. Rev. D* **77**, 016002 (2008), arXiv:0709.2502; (b) F.R. Klinkhamer and M. Risse, “Addendum: Ultrahigh-energy cosmic-ray bounds on nonbirefringent modified-Maxwell theory,” *Phys. Rev. D* **77**, 117901 (2008), arXiv:0709.2502; (c) F.R. Klinkhamer and M. Schreck, “New two-sided bound on the isotropic Lorentz-violating parameter of modified Maxwell theory,” *Phys. Rev. D* **78**, 085026 (2008), arXiv:0809.3217.
- [9] M.J. Duff and P. van Nieuwenhuizen, “Quantum inequivalence of different field representations,” *Phys. Lett. B* **94**, 179 (1980).
- [10] A. Aurilia, H. Nicolai, and P.K. Townsend, “Hidden constants: The theta parameter of QCD and the cosmological constant of N=8 supergravity,” *Nucl. Phys. B* **176**, 509 (1980).
- [11] G.E. Volovik, “Cosmological constant and vacuum energy,” *Ann. Phys. (Leipzig)* **14**, 165 (2005), arXiv:gr-qc/0405012.
- [12] G.E. Volovik, “Evolution of cosmological constant in effective gravity,” *JETP Lett.* **77**, 339 (2003), arXiv:gr-qc/0302069.
- [13] C. Barcelo, “Cosmology as a search for overall equilibrium,” *JETP Lett.* **84**, 635 (2007), arXiv:gr-qc/0611090.
- [14] L. Amendola, G. Camargo Campos, and R. Rosenfeld, “Consequences of dark matter–dark energy interaction on cosmological parameters derived from SNIa data,” *Phys. Rev. D* **75**, 083506 (2007), arXiv:astro-ph/0610806.
- [15] R. Penrose, “Singularities and time-asymmetry,” in: *General Relativity: An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979), Chap. 12.
- [16] F.R. Klinkhamer, “Fundamental time asymmetry from nontrivial space topology,” *Phys. Rev. D* **66**, 047701 (2002), arXiv:gr-qc/0111090.
- [17] (a) A. Vilenkin, “Quantum creation of universes,” *Phys. Rev. D* **30**, 509 (1984); (b) A. Vilenkin, “Boundary conditions in quantum cosmology,” *Phys. Rev. D* **33**, 3560 (1986); (c) A. Vilenkin, “The quantum cosmology debate,” arXiv:gr-qc/9812027.
- [18] B. Ratra and P.J.E. Peebles, “Cosmological consequences of a rolling homogeneous scalar field,” *Phys. Rev. D* **37**, 3406 (1988).
- [19] D.J. Eisenstein *et al.* [SDSS Collaboration], “Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies,” *Astrophys. J.* **633**, 560 (2005), arXiv:astro-ph/0501171.

- [20] P. Astier *et al.* [SNLS Collaboration], “The Supernova Legacy Survey: Measurement of Ω_M , Ω_Λ and w from the first year data set,” *Astron. Astrophys.* **447**, 31 (2006), arXiv:astro-ph/0510447.
- [21] A.G. Riess *et al.*, “New Hubble Space Telescope discoveries of type Ia supernovae at $z > 1$: Narrowing constraints on the early behavior of dark energy,” *Astrophys. J.* **659**, 98 (2007), astro-ph/0611572.
- [22] E. Komatsu *et al.*, “Five-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Cosmological interpretation,” arXiv:0803.0547v1.
- [23] W.L. Freedman *et al.* [Hubble Space Telescope Collaboration], “Final results from the Hubble Space Telescope Key Project to measure the Hubble constant,” *Astrophys. J.* **553**, 47 (2001), arXiv:astro-ph/0012376.
- [24] V. Sahni and A.A. Starobinsky, “Reconstructing dark energy,” *Int. J. Mod. Phys. D* **15**, 2105 (2006), arXiv:astro-ph/0610026.
- [25] F.R. Klinkhamer and G.E. Volovik, “Dynamic vacuum variable and equilibrium approach in cosmology,” *Phys. Rev. D* **78**, 063528 (2008), arXiv:0806.2805.
- [26] F.R. Klinkhamer and G.E. Volovik, “ $f(R)$ cosmology from q -theory,” *JETP Lett.* **88**, 289 (2008), arXiv:0807.3896.